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RECURRENCE RELATIONS OF COEFFICIENTS OF THE GENERALIZED HYPERGEOMETRIC FUNCTION AND THE ZONAL POLYNOMIAL

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1. Introduction

James (1960) obtained the joint distributions of the characteristic roots of the covariance matrix on Wishart matrix by constituting the zonal polynomials. It was epoch-making in the history of multivariate distribution theory. Since that, the many density functions and moments in multivariate analysis have been expressed by the zonal polynomials and the generalized hypergeometric function of symmetric matrix argument. We may see some results in James (1964). Examples are the noncentral distributions of the characteristic roots in multiple discriminant analysis (Constantine (1963)), the distributions of the largest characteristic root and the corresponding characteristic vector of a Wishart matrix (Sugiyama (1966,1967)) and the distributions of the largest and smallest root of a Wishart distribution of a multivariate beta distribution (Constantine (1963)).

The distributions expressed by the generalized hypergeometric function may be written by the zonal polynomial series. Generally speaking, they converge extremely slowly, and the methods for approximating them have received a great deal of attention (see Fujikoshi (1968), (1970), (1973), Sugiura (1969), (1972), (1974) and Sugiura and Fujikoshi (1969)). Actually, the asymptotic distributions and moments are very useful in practical points of view. But to know the accuracy on the various sample sizes and population parameters it is necessary to compute the exact distributions and moments. We derive the partial differential equation to obtain the recurrence relations of coefficients of the generalized hypergeometric function for their computations.

Sugiyama (1979) has obtained the coefficients of zonal polynomials up to degree 200 in the case of order 2 and the programming to compute it, expressed by a linear combination of monomial symmetric function. At the practical point of view, it is very important that the cumulative density function, moments, and so on are possible to compute in the case of order 3, and also higher order. So we obtain an algorithm based on the recurrence relation due to James.

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2. Recurrence relations of coefficients of the generalized hypergeometric function in multivariate analysis

2.1. Partial differential equations for the generalized hypergeometric function.

The following lemma in Muirhead (1982) shows that the generalized hypergeometric function ${}_2F_1(a, b; c; Y)$ having the form

$$F = \sum_{k=0}^{\infty} \sum_{\kappa} \alpha_\kappa C_\kappa(Y) \qquad (\alpha_{(0)} = 1),$$

is the unique solution of partial differential equations, where $y_1, \cdots, y_m$ are the characteristic roots of the symmetric matrix Y and $C_\kappa(Y)$ are defined for each partition $\kappa = (k_1, \ldots, k_m)$, $k_1 \geq \ldots \geq k_m \geq 0$, of k into not more than m parts.

Lemma 2.1 The function ${}_2F_1(a, b; c; Y)$ is the unique solution of each of the m partial differential equations

$$\begin{aligned} y_i(1-y_i)\frac{\partial^2 F}{\partial y_i^2} &+ \left\{ c - \frac{1}{2}(m-1) \right. \\ &- \left. \left[a+b+1-\frac{1}{2}(m-1)\right] y_i + \frac{1}{2}\sum_{\substack{j=1 \\ j\neq i}}^{m} \frac{y_i(1-y_i)}{y_i-y_j} \right\} \frac{\partial F}{\partial y_i} \\ &- \frac{1}{2}\sum_{\substack{j=1 \\ j\neq i}}^{m} \frac{y_j(1-y_j)}{y_i-y_j}\frac{\partial F}{\partial y_j} = abF, \qquad (i = 1, \ldots, m) \end{aligned} \tag{2.1}$$

subject to the conditions that

(a) F is a symmetric function of $y_1, \ldots, y_m$, and

(b) F is analytic at $Y = 0$, and $F(0) = 1$.

We show that the m equations (2.1) have the same unique solution subject to the conditions (a) and (b) by transforming to a system of equations in terms of the elementary symmetric functions $r_1 = \sum_{i=1}^m y_i$, $r_2 = \sum_{i<j}^m y_i y_j$, $\ldots$, $r_m = y_1 y_2 \cdots y_m$. Let $r_j^{(i)}$ for $j = 1, 2, \ldots, m-1$ denote the jth elementary symmetric function formed from $y_1, \ldots, y_m$ omitting y_i.

Introducing the dummy variables

$$\begin{aligned} r_0 &= r_0^{(i)} = 1 \\ r_j &= r_j^{(i)} = 0 \qquad \text{for } j < 0,\ j > m. \end{aligned}$$

we have the relationship

$$r_j = y_i r_{j-1}^{(i)} + r_j^{(i)} \qquad (-\infty < j < \infty).$$

It follows that

$$\begin{aligned} \frac{\partial F}{\partial y_i} &= \sum_{\nu=1}^{m} r_{\nu-1}^{(i)} \frac{\partial F}{\partial r_\nu} \\ \frac{\partial^2 F}{\partial y_i^2} &= \sum_{\mu,\nu=1}^{m} r_{\mu-1}^{(i)} r_{\nu-1}^{(i)} \frac{\partial^2 F}{\partial r_\mu \partial r_\nu}. \end{aligned}$$

Substituting these in (2.1), the system becomes

$$\begin{aligned}
& y_i(1-y_i)\sum_{\mu,\nu=1}^{m} r^{(i)}_{\mu-1} r^{(i)}_{\nu-1} \frac{\partial^2 F}{\partial r_\mu \partial r_\nu} + \Bigg\{ c - \frac{1}{2}(m-1) \\
& - \left[a+b+1-\frac{1}{2}(m-1)\right] y_i + \frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{y_i(1-y_i)}{y_i-y_j} \Bigg\} \sum_{\nu=1}^{m} r^{(i)}_{\nu-1}\frac{\partial F}{\partial r_\nu} \\
& -\frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{y_j(1-y_j)}{y_i-y_j} \sum_{\nu=1}^{m} r^{(j)}_{\nu-1}\frac{\partial F}{\partial r_\nu} = abF, \qquad (i=1,\dots,m).
\end{aligned} \tag{2.2}$$

Now we have

$$\begin{aligned}
y_i r^{(i)}_{\nu-1} r^{(i)}_{\mu-1} &= r_\nu r^{(i)}_{\mu-1} - r^{(i)}_\nu r^{(i)}_{\mu-1} \\
&= r_\nu r^{(i)}_{\mu-1} - r^{(i)}_\nu (r_{\mu-1} - y_i r^{(i)}_{\mu-2}) \\
&= r_\nu r^{(i)}_{\mu-1} - r^{(i)}_\nu r_{\mu-1} + y_i r^{(i)}_\nu r^{(i)}_{\mu-2} \\
&= \cdots \\
&= \sum_{j=1}^{\mu} r_{\nu+\mu-j} r^{(i)}_{j-1} - \sum_{j=1}^{\mu} r_{\mu-j} r^{(i)}_{\nu+j-1} \\
&= \sum_{j=1}^{\mu} r_{\nu+\mu-j} r^{(i)}_{j-1} - \sum_{j=\nu+1}^{\mu+\nu} r_{\nu+\mu-j} r^{(i)}_{j-1} \\
&= \sum_{j=1}^{\min(\mu,\nu)} r_{\nu+\mu-j} r^{(i)}_{j-1} - \sum_{j=\max(\mu,\nu)+1}^{\mu+\nu} r_{\nu+\mu-j} r^{(i)}_{j-1},
\end{aligned}$$

and from $r_j = y_i r^{(i)}_{j-1} + r^{(i)}_j$,

$$1 - y_i = \frac{r^{(i)}_{j-1} - r_j + r^{(i)}_j}{r^{(i)}_{j-1}}.$$

Then the first term of (2.2) is written as

$$\begin{aligned}
& \sum_{\mu,\nu=1}^{m} \left(\sum_{j=1}^{\min(\mu,\nu)} r_{\nu+\mu-j}(r^{(i)}_{j-1} - r_j + r^{(i)}_j) - \sum_{j=\max(\mu,\nu)+1}^{\mu+\nu} r_{\nu+\mu-j}(r^{(i)}_{j-1} - r_j + r^{(i)}_j) \right) \frac{\partial^2 F}{\partial r_\mu \partial r_\nu} \\
& \qquad = \sum_{\mu,\nu=1}^{m}\sum_{j=1}^{m} a^{(j)}_{\mu\nu} \left(r^{(i)}_{j-1} - r_j + r^{(i)}_j \right) \frac{\partial^2 F}{\partial r_\mu \partial r_\nu},
\end{aligned}$$

where $a^{(j)}_{\mu\nu} = a^{(j)}_{\nu\mu}$ and, for $\mu \le \nu$,

$$a^{(j)}_{\mu\nu} = \begin{cases} r_{\mu+\nu-j} & \text{for } 1 \le j \le \mu \\ 0 & \text{for } \mu < j \le \nu \\ -r_{\mu+\nu-j} & \text{for } \nu < j \le \mu+\nu \\ 0 & \text{for } \mu+\nu < j. \end{cases}$$

Also we may rewrite as follows

$$\sum_{\substack{j=1\\j\neq i}}^{m}\frac{y_i(1-y_i)}{y_i-y_j}r^{(i)}_{\nu-1}-\sum_{\substack{j=1\\j\neq i}}^{m}\frac{y_j(1-y_j)}{y_i-y_j}r^{(j)}_{\nu-1}$$
$$=\sum_{\substack{j=1\\j\neq i}}^{m}\frac{r_\nu-r^{(i)}_\nu}{y_i-y_j}(1-y_i)-\sum_{\substack{j=1\\j\neq i}}^{m}\frac{r_\nu-r^{(j)}_\nu}{y_i-y_j}(1-y_j)$$
$$=(m-\nu)r^{(i)}_{\nu-1}-(m-1)r_\nu+(m-\nu-1)r^{(i)}_\nu.$$

Applying these equations, the system (2.2) is written as

$$\sum_{\mu,\nu=1}^{m}\sum_{j=1}^{m}a^{(j)}_{\mu\nu}\left(r^{(i)}_{j-1}-r_j+r^{(i)}_j\right)\frac{\partial^2F}{\partial r_\mu\partial r_\nu}$$
$$+\sum_{j=1}^{m}\left\{\left[c-\frac{1}{2}(j-1)\right]r^{(i)}_{j-1}+\left(a+b+1-\frac{1}{2}j\right)r^{(i)}_j-(a+b+1)r_j\right\}$$
$$\cdot\frac{\partial F}{\partial r_j}-abF=0,\qquad(i=1,\dots,m).\tag{2.3}$$

Any solution of (2.3) satisfies the condition (a). In (2.3) we may equate the coefficients of $r^{(i)}_{j-1}$ to zero for $j=1,\dots,m$ according to the following Lemma due to James (1955).

Lemma 2.2 If $y_1,\dots,y_m$ are the characteristic roots of the symmetric matrix Y and $r_1,\dots,r_m$ are the elementary symmetric functions of them, and $r^{(i)}_1,\dots,r^{(i)}_{m-1}$ the elementary symmetric functions of $y_1,\dots,y_m$ with y_i omitted and if $\lambda_0(r)$, $\lambda_1(r),\dots,$ $\lambda_{m-1}(r)$ are functions of $r_1,\dots,r_m$ such that

$$\lambda_0(r)+\lambda_1(r)\,r^{(i)}_1+\lambda_2(r)\,r^{(i)}_2+\cdots+\lambda_{m-1}(r)\,r^{(i)}_{m-1}=0,$$

then

$$\lambda_0(r)=0,\quad\lambda_1(r)=0,\quad\dots\quad,\quad\lambda_{m-1}(r)=0.$$

Equating coefficients of $r^{(i)}_{j-1}$ in (2.3) to zero we have the system

$$\sum_{\mu,\nu=1}^{m}\left(a^{(j)}_{\mu\nu}+a^{(j-1)}_{\mu\nu}\right)\frac{\partial^2F}{\partial r_\mu\partial r_\nu}+\left[c-\frac{1}{2}(j-1)\right]\frac{\partial F}{\partial r_j}$$
$$+\left[a+b+1-\frac{1}{2}(j-1)\right]\frac{\partial F}{\partial r_{j-1}}-\delta_{1j}\left\{\sum_{\mu,\nu=1}^{m}\sum_{\ell=1}^{m}a^{(\ell)}_{\mu\nu}r_\ell\frac{\partial^2F}{\partial r_\mu\partial r_\nu}\right.$$
$$\left.+(a+b+1)\sum_{\ell=1}^{m}r_\ell\frac{\partial F}{\partial r_\ell}+abF\right\}=0,\qquad(j=1,\dots,m),\tag{2.4}$$

where δ_{1j} is the Kronecker delta. Now we put

$$F(r_1,\dots,r_m)=\sum_{j_1,\dots,j_m=0}^{\infty}\gamma(j_1,\dots,j_m)r_1^{j_1}\cdots r_m^{j_m}\tag{2.5}$$

where $\gamma(0,\ldots,0)=1$. We introduce dictionary ordering for the coefficients $\gamma(j_1,\ldots,j_m)$ on the basis of the indices arranged in the order $j_m, j_{m-1},\ldots,j_2,j_1$. Substituting (2.5) in (2.4) with $j=m$ gives a recurrence relation which expresses $\gamma(j_1,\ldots,j_m)$ in terms of coefficients whose last index is less than j_m, and by iteration $\gamma(j_1,\ldots,j_m)$ can be expressed in terms of coefficients whose last index is zero. Putting $r_m=0$ in the equation (2.4) with $j=m-1$, then we can express coefficients of the form $\gamma(j_1,\ldots,j_{m-1},0)$ in terms of coefficients of the form $\gamma(j_1,\ldots,j_{m-2},0,0)$. By repeating this procedure, we may obtain the recurrence relations, where $\gamma(0,\ldots,0)$ is 1. Hence all the coefficients $\gamma(j_1,\ldots,j_m)$ in (2.5) are uniquely determined by the recurrence relations, and condition (b) is satisfied. Since each differential equation in (2.1) gives rise to the same system (2.4), it follows that each equation in the system (2.1) has the same unique solution F subject to conditions (a) and (b).

Now, the series (2.5) could be rearranged as a series of zonal polynomials

$$F=\sum_{k=0}^{\infty}\sum_{\kappa}\alpha_\kappa C_\kappa(Y), \qquad \alpha_{(0)}=1. \tag{2.6}$$

Since the zonal polynomials expressed in terms of the elementary symmetric functions $r_1,\ldots,r_m$ do not explicitly depend on m, the coefficients α_κ can be functions of a,b,c and κ but not m. Since $C_\kappa(Y)\equiv 0$ for any partition into more than m nonzero parts, the α_κ can be defined for partitions into any number of parts and are completely independent of m. Hence the unique solution of (2.1) subject to (a) and (b) can be expressed as (2.6), where the coefficients α_κ are independent of m.

2.2. Recurrence relations for coefficients of ${}_2F_1$ function, ${}_1F_1$ function and of ${}_0F_1$ function

In this section we obtain the recurrence relations for coefficients of multiple power series of the elementary symmetric functions, and by iterating this relation we evaluate the hypergeometric functions of matrix argument.

In the differential equation (2.4) with $j=3$ and $r_4=\cdots=r_m=0$, we have the following

$$\begin{aligned}-2\frac{\partial^2 F}{\partial r_1\partial r_2}-r_1\frac{\partial^2 F}{\partial r_2^2}+r_3\frac{\partial^2 F}{\partial r_3^2}-\frac{\partial^2 F}{\partial r_1^2}+r_2\frac{\partial^2 F}{\partial r_2^2}&\\ +2r_3\frac{\partial^2 F}{\partial r_2\partial r_3}+(c-1)\frac{\partial F}{\partial r_3}+(a+b)\frac{\partial F}{\partial r_2}&=0.\end{aligned} \tag{2.7}$$

On substituting

$$F=\sum_{j_1,j_2,j_3=0}^{\infty}\gamma(j_1,j_2,j_3,\boldsymbol{o})r_1^{j_1}r_2^{j_2}r_3^{j_3}, \tag{2.8}$$

where $\boldsymbol{o}$ is $(0,\ldots,0)$ with $m-3$ components, in these differential equations and equating coefficients of $r_1^{j_1}r_2^{j_2}r_3^{j_3}$, we have the recurrence relation

$$\begin{aligned}\gamma(j_1,j_2,j_3,\boldsymbol{o}) \;=\; & \{2(j_1+1)(j_2+1)\gamma(j_1+1,j_2+1,j_3-1,\boldsymbol{o})\\ & +(j_2+1)(j_2+2)\gamma(j_1-1,j_2+2,j_3-1,\boldsymbol{o})\\ & +(j_1+1)(j_1+2)\gamma(j_1+2,j_2,j_3-1,\boldsymbol{o})\\ & -(j_2+1)(j_2+2j_3+a+b-2)\gamma(j_1,j_2+1,j_3-1,\boldsymbol{o})\}\\ & \cdot\{j_3(j_3+c-2)\}^{-1},\end{aligned}$$

which can be iterated to express $\gamma(j_1,j_2,j_3,\boldsymbol{o})$ in terms of coefficients of the form $\gamma(j_1,j_2,\boldsymbol{o})$. When $j=2$ and $r_3=\cdots=r_m=0$, we have the relation

$$\begin{aligned}\gamma(j_1,j_2,\boldsymbol{o}) \;=\; & \{(j_1+1)(j_1+2)\gamma(j_1+2,j_2-1,\boldsymbol{o})\\ & -(j_1+1)\left(j_1+2j_2+a+b-\frac{3}{2}\right)\gamma(j_1+1,j_2-1,\boldsymbol{o})\Big\}\\ & \cdot\left\{j_2\left(j_2+c-\frac{3}{2}\right)\right\}^{-1}\end{aligned}$$

When $j=1$ and $r_2=\cdots=r_m=0$, we have the recurrence relation

$$\gamma(j_1,\boldsymbol{o})=\{(j_1-1)(a+b+j_1-1)+ab\}\gamma(j_1-1,\boldsymbol{o})\{j_1(c+j_1-1)\}^{-1}.$$

In the case $m=3$, using above three equations, we may evaluate ${}_2F_1$ function of 3×3 matrix argument by computer algorithm. The recurrence relations for coefficients of multiple power series of the elementary symmetric functions for ${}_2F_1$ function will be given up to $j=7$ in Appendix A.1.

Lemma 2.1 also yields the systems of partial differential equations satisfied by the ${}_1F_1$ and ${}_0F_1$ functions. So, by the same procedures as above we obtain

$$\begin{aligned}&\sum_{\mu,\nu=1}^{m}\left(\sum_{j=1}^{\mu}r_{\nu+\mu-j}r_{j-1}^{(i)}-\sum_{j=1}^{\mu}r_{\mu-j}r_{\nu+j-1}^{(i)}\right)\frac{\partial^2 F}{\partial r_\mu\partial r_\nu}+\sum_{j=1}^{m}\left\{\left[c-\frac{1}{2}(j-1)\right]r_{j-1}^{(i)}\right.\\ &\qquad\qquad\left.+r_j^{(i)}-r_j\right\}\frac{\partial F}{\partial r_j}-aF=0,\qquad (i=1,\dots,m)\end{aligned}\tag{2.9}$$

for ${}_1F_1(a;c;Y)$ function, and

$$\begin{aligned}&\sum_{\mu,\nu=1}^{m}\left(\sum_{j=1}^{\mu}r_{\nu+\mu-j}r_{j-1}^{(i)}-\sum_{j=1}^{\mu}r_{\mu-j}r_{\nu+j-1}^{(i)}\right)\frac{\partial^2 F}{\partial r_\mu\partial r_\nu}\\ &\quad+\sum_{j=1}^{m}\left[c-\frac{1}{2}(j-1)\right]r_{j-1}^{(i)}\frac{\partial F}{\partial r_j}-F\;=\;0,\quad (i=1,\dots,m)\end{aligned}\tag{2.10}$$

for ${}_0F_1(c;Y)$ function. Equating coefficients of $r_{j-1}^{(i)}$ to zero for $j=1,\dots,m$, then we have the system

$$\begin{aligned}&\sum_{\mu,\nu=1}^{m}a_{\mu\nu}^{(j)}\frac{\partial^2 F}{\partial r_\mu\partial r_\nu}+\left[c-\frac{1}{2}(j-1)\right]\frac{\partial F}{\partial r_j}\\ &\quad+\frac{\partial F}{\partial r_{j-1}}-\delta_{1j}\left(\sum_{\ell=1}^{m}r_\ell\frac{\partial F}{\partial r_\ell}+aF\right)=0,\qquad (j=1,\dots,m)\end{aligned}\tag{2.11}$$

for ${}_1F_1$ functions, and

$$\sum_{\mu,\nu=1}^{m} a_{\mu\nu}^{(j)} \frac{\partial^2 F}{\partial r_\mu \partial r_\nu} + \left[c - \frac{1}{2}(j-1)\right] \frac{\partial F}{\partial r_j} - \delta_{1j} F = 0, \qquad (j = 1, \ldots, m) \tag{2.12}$$

for ${}_0F_1$ function. Substituting

$$F(r_1, \ldots, r_m) = \sum_{j_1,\cdots,j_m=0}^{\infty} \gamma(j_1, \ldots, j_m) r_1^{j_1} \cdots r_m^{j_m} \tag{2.13}$$

with $\gamma(0, \ldots, 0) = 1$ in the equations, we may obtain the recurrence relations of γ. The recurrence relations for coefficients of multiple power series of the elementary symmetric functions for ${}_1F_1$ and ${}_0F_1$ function will be given up to $j = 7$ in Appendix A.2 and Appendix A.3.

Remark: To know the exact values of the generalized hypergeometric function ${}_2F_1$ or ${}_1F_1$ expressed by the formula (2.6), we have to compute up to $k = 200$, 250, and so on. So, at the present computers it may be possible to compute at most until $j = 4$.

2.3. Numerical analysis of the largest characteristic root in the principal components analysis

Let $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N$ be a sample from $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and ℓ_1 be the largest characteristic root of the Wishart matrix corresponding to sample size $N = n + 1$. The distribution of ℓ_1 is given by Sugiyama as follows

$$F(\ell_1 < nx) = \text{Const.} \cdot \exp\left(-\frac{nx}{2}\mathrm{tr}\boldsymbol{\Sigma}^{-1}\right) (nx)^{\frac{pn}{2}} \cdot {}_1F_1\left(\frac{p+1}{2}; \frac{n+p+1}{2}; \frac{nx}{2}\boldsymbol{\Sigma}^{-1}\right), \tag{2.14}$$

where

$$\text{Const.} = |\boldsymbol{\Sigma}|^{-\frac{n}{2}} \Gamma_p(\frac{p+1}{2})/2^{\frac{np}{2}} \Gamma_p(\frac{n+p+1}{2}),$$

and

$${}_1F_1(a; c; \boldsymbol{Y}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a)_\kappa}{(c)_\kappa} \frac{C_\kappa(\boldsymbol{Y})}{k!}.$$

When $\boldsymbol{\Sigma} = \boldsymbol{I}$, the c.d.f. of the largest characteristic root ℓ_1 may be written as follows

$$F(\ell_1 < nx) = \frac{\Gamma_p(\frac{p+1}{2})}{2^{\frac{np}{2}} \Gamma_p(\frac{n+p+1}{2})} \cdot \exp\left(-\frac{npx}{2}\right) (nx)^{\frac{pn}{2}} \cdot {}_1F_1\left(\frac{p+1}{2}; \frac{n+p+1}{2}; \frac{nx}{2}\boldsymbol{I}\right). \tag{2.15}$$

By the quadruple precision arithmetic, we obtain the upper 5% points for the sample size $n + 1$. On the other hand, Sugiyama obtained the c.d.f. using the explicit formula of the zonal polynomial $C_\kappa(\boldsymbol{I})$. Table 2.1 shows the accuracy of the upper 5% points on the formula (2.14).

When $\boldsymbol{\Sigma} = diag(2.0, 1.2, 0.8)$ we obtained the values of the upper 5% points. In this case we do not know the exact values of the percentile points. So, to see the accuracy, we did the simulation study. With respect to values obtained by the one million simulation it may be expected for at most 3 figures to be reliable.

The convergence of the hypergeometric function become slow when x is large. And also it depends on the values of the characteristic roots of $\frac{nx}{2}\boldsymbol{\Sigma}^{-1}$. So if the sample size $n+1$ is larger, the convergence becomes slower. It means that the numerical computation may be more difficult.

Table 2.1 the upper 5% points for $\boldsymbol{\Sigma} = \boldsymbol{I}$

n	recurrence relation result	Sugiyama result
2	5.370173	5.370173
4	3.810174	3.810174
6	3.181457	3.181457
8	2.828000	2.828000
10	2.596608	2.596608
12	2.431132	2.431132
14	2.305742	2.305741
16	2.206759	2.206759
18	2.126207	2.126207
20	2.059093	2.059093
22	2.002116	2.002116

Table 2.2 the upper 5% points for $\boldsymbol{\Sigma} = diag(2.0, 1.2, 0.8)$

n	recurrence relation result	simulation result
2	7.646561	7.65
4	5.602895	5.61
6	4.779797	4.78
8	4.318502	4.32
10	4.017546	4.02
12	3.803020	3.81
14	3.640940	3.64
16	3.514184	3.52

$n = N - 1$

3. Algorithm on recurrence relations between coefficients of zonal polynomials

3.1. Zonal polynomials

The zonal polynomials are an eigenfunction of the Laplace Beltrami operator,

$$\Delta = (\det G)^{-\frac{1}{2}} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} (\det G)^{\frac{1}{2}} \sum_{i=1}^{n} g^{ik} \frac{\partial}{\partial x_i}, \tag{3.1}$$

where $x_1, \cdots, x_n$ are coordinates of a point in a space with metric differential form

$$(ds)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} dx_i dx_j,$$

$$G = (g_{ij}) \quad \text{and} \quad (g^{ij}) = G^{-1}.$$

On substituting the part of the Laplace Beltrami operator concerned with the roots, we have the operator

$$\Delta = \sum_{i=1}^{m} \left[y_i^2 \frac{\partial^2}{\partial y_i^2} - \frac{1}{2}(m-3) y_i \frac{\partial}{\partial y_i} \right] + \sum_{\substack{i,\, j\, =\, 1 \\ i \neq j}}^{m} \frac{y_i^2}{y_i - y_j} \frac{\partial}{\partial y_i}. \tag{3.2}$$

In detail we may see in James (1968).

The zonal polynomials of a matrix are defined in terms of partitions of positive integers. Let k be a positive integer. A partition κ of k is written as $\kappa = (k_1, k_2, \ldots)$, where $\sum_i k_i = k$. We order the partitions of k lexicographically, that is, if $\kappa = (k_1, k_2, \ldots)$ and $\lambda = (l_1, l_2, \ldots)$ are two partitions of k, we write $\kappa > \lambda$ if $k_i > l_i$ for the first index i for which the parts are unequal. Let $y_1, \cdots, y_m$ be m variables. If $\kappa > \lambda$ we say that the monomial $y_1^{k_1} \cdots y_m^{k_m}$ is of higher weight than the monomial $y_1^{l_1} \cdots y_m^{l_m}$. In this section, the zonal polynomials defined by monomial symmetric function $M_\kappa(Y)$ have an important role

$$M_\kappa(Y) = y_1^{k_1} y_2^{k_2} \cdots y_m^{k_m} + \text{symmetric terms}. \tag{3.3}$$

Let Y be an $m \times m$ symmetric matrix with the characteristic roots $y_1, \cdots, y_m$ and let $\kappa = (k_1, \cdots, k_m)$ be a partition of k into not more than m parts. The zonal polynomial of Y corresponding to κ, denoted by $C_\kappa(Y)$, is a symmetric homogeneous polynomial of degree k in the characteristic roots $y_1, \cdots, y_m$ such that;

(i) The term of the highest weight in $C_\kappa(Y)$ is $y_1^{k_1} \cdots y_m^{k_m}$, that is

$$C_\kappa(Y) = c_{\kappa,\kappa} y_1^{k_1} \cdots y_m^{k_m} + \text{terms of lower weight}, \tag{3.4}$$

where $c_{\kappa,\kappa}$ is a constant.

(ii) $C_\kappa(Y)$ is an eigenfunction of the differential operator Δ_Y given by

$$\Delta_Y = \sum_{i=1}^{m} y_i^2 \frac{\partial^2}{\partial y_i^2} + \sum_{\substack{i,\, j\, =\, 1 \\ i \neq j}}^{m} \frac{y_i^2}{y_i - y_j} \frac{\partial}{\partial y_i}. \tag{3.5}$$

(iii) As κ varies over all partitions of k the zonal polynomials have unit coefficients in the expansion of $(\text{tr } Y)^k$, that is

$$(\text{tr } Y)^k = (y_1 + y_2 + \cdots + y_m)^k = \sum_{\kappa} C_\kappa(Y). \tag{3.6}$$

So the zonal polynomial of Y corresponding to the partition κ satisfies the partial differential equation

$$\Delta_Y C_\kappa(Y) = [\rho_\kappa + k(m-1)]C_\kappa(Y), \tag{3.7}$$

where $\rho_\kappa = \sum_{i=1}^m k_i(k_i - i)$. On substituting the formula (3.4) of the zonal polynomial we have the recurrence relation between the coefficients $c_{\kappa,\lambda}$,$c_{\kappa,\mu}$ of the monomial symmetric functions $M_\lambda(Y)$, $M_\mu(Y)$ as follows

$$c_{\kappa,\lambda} = \sum_{\lambda<\mu\leq\kappa} \frac{(l_i + r) - (l_j - r)}{\rho_\kappa - \rho_\lambda} c_{\kappa,\mu} \tag{3.8}$$

where

$$\rho_\lambda = \sum_{i=1}^m l_i(l_i - i), \qquad \lambda = (l_1, \cdots, l_m),$$

and

$$\mu = (l_1, \cdots, l_i + r, \cdots, l_j - r, \cdots, l_m).$$

The recurrence relation determines the zonal polynomial uniquely except a normalizing constant. It is given by obtaining the coefficient of the term of the highest weight, according to the following property

$$C_\kappa(Y) = \chi_{[2\kappa]}(1)\frac{2^k k!}{(2k)!} Z_\kappa(Y), \tag{3.9}$$

where $Z_\kappa(Y)$ is the zonal polynomials with a different normalizing constant, and

$$\chi_{[\kappa]}(1) = k! \prod_{i<j}^m (k_i - k_j - i + j) \Big/ \prod_{i=1}^m (k_i + m - i)!.$$

The coefficient of the monomial of the highest weight $y_1^{k_1} y_2^{k_2} \cdots y_m^{k_m}$ in $Z_\kappa(Y)$ is

$$2^k \prod_{l=1}^p \prod_{i=1}^l \left(\frac{1}{2} l - \frac{1}{2}(i-1) + k_i - k_l \right)_{k_l - k_{l+1}} \tag{3.10}$$

where p is the number of non zero parts in the partition and $(a)_k = a(a+1)\cdots(a+k-1)$. The zonal polynomials are expressed by the monomial symmetric function as follows

$$C_\kappa(Y) = c_{\kappa,\kappa} M_\kappa(Y) + \text{terms of lower weight}. \tag{3.11}$$

Suppose that κ is a partition of k into p non zero parts. If the partition λ of k has less than p non zero parts and $\lambda < \kappa$, then $c_{\kappa,\lambda} = 0$ (see Muirhead (1982)). So we have the following lemma.

Lemma 3.1 Let $\kappa = (k_1, k_2, \cdots, k_{m-1}, k_m)$ be a partition of k, and $\kappa' = (k_1 - k_m, k_2 - k_m, \cdots, k_{m-1} - k_m, 0)$ be a partition of $k - mk_m$. Then we the have the following equation

$$C_\kappa(Y) = a_{\kappa'} C_{\kappa'}(Y)(y_1 \cdots y_m)^{k_m}, \tag{3.12}$$

where $a_{\kappa'}$ is constant, and

$$a_\kappa = \frac{C_\kappa(I_m)}{C_{\kappa'}(I_m)} \tag{3.13}$$

where

$$C_\kappa(I_m) = 2^{2k} k! (\tfrac{1}{2}m)_\kappa \prod_{i<j}^{m} (2k_i - 2k_j - i + j) \Big/ \prod_{i=1}^{m} (2k_i + m - i)!. \tag{3.14}$$

Theorem 3.1 *Let Y be an $m \times m$ symmetric matrix with the characteristic roots $y_1, \cdots, y_m$ and a partition $\kappa = (k_1, \cdots, k_m)$ of k. The coefficients of the zonal polynomial corresponding to κ satisfy the recurrence relation* (3.8) *and the coefficient of the term of the highest weight is given by*

$$c_{\kappa,\kappa} = \chi_{[2\kappa]}(1) \frac{2^{2k} k!}{(2k)!} \prod_{l=1}^{p} \prod_{i=1}^{l} \left(\frac{1}{2} l - \frac{1}{2}(i-1) + k_i - k_l \right)_{k_l - k_{l+1}}. \tag{3.15}$$

In the case of order 3, the recurrence relation of the coefficients of the zonal polynomial $C_\kappa(Y)$ are given by the following equations

$$c_{\kappa,\lambda} = \sum_{\lambda < \mu \leq \kappa} \frac{(l_i + r) - (l_j - r)}{\rho_\kappa - \rho_\lambda} c_{\kappa,\mu} \tag{3.16}$$

where

$$\rho_\lambda = \sum_{i=1}^{3} l_i (l_i - i), \qquad \lambda = (l_1, l_2, l_3),$$

and

$$\mu = \begin{cases} (\ l_1 + r,\ l_2 - r,\ \ l_3\ ) \\ (\ l_1 + r,\ \ l_2,\ \ l_3 - r\ ) \\ (\ \ l_1,\ \ l_2 + r,\ l_3 - r\ ). \end{cases}$$

3.2. Representations of zonal polynomials

We know three types representations of the zonal polynomials. For example, we show the case of order 3 in the following table

Table 3.1 Three types representations of the zonal polynomials.

	In terms of monomial symmetric functions of the characteristic roots of Y $b_1 = y_1^3 + y_2^3 + y_3^3$ $b_2 = \sum_{i \neq j} y_i^2 y_j$ $b_3 = y_1 y_2 y_3$	In terms of sums of powers of the characteristic roots of Y $s_1 = y_1 + y_2 + y_3$ $s_2 = y_1^2 + y_2^2 + y_3^3$ $s_3 = y_1^3 + y_2^3 + y_3^3$	In terms of elementary symmetric functions of the characteristic roots of Y $a_1 = y_1 + y_2 + y_3$ $a_2 = y_1 y_2 + y_2 y_3 + y_3 y_1$ $a_3 = y_1 y_2 y_3$
$Z_{(3,0,0)}$	$15b_1 + 9b_2 + 6b_3$	$s_1^3 + 6s_1 s_2 + 8s_3$	$15a_1^3 - 36a_1 a_2 + 24a_3$
$Z_{(2,1,0)}$	$4b_2 + 6b_3$	$s_1^3 + s_1 s_2 - 2s_3$	$4a_1 a_2 - 6a_3$
$Z_{(1,1,1)}$	$6b_3$	$s_1^3 - 3s_1 s_2 + 2s_3$	$6a_3$

Obviously, in Table 3.1 $b_1 = M_{(3,0,0)}$, $b_2 = M_{(2,1,0)}$ and $b_3 = M_{(1,1,1)}$. We notice that the coefficients of the zonal polynomials are positive, in the case of monomial symmetric function. This is true for arbitrary k and the partition κ. When we compute the functions expressed by the zonal polynomials, it is a big advantage that we do not compute any subtraction, because we lose a lot of significant figures by the subtraction. Therefore we obtain the Fortran Program to get the zonal polynomials by monomial symmetric function. Then we have the coefficient of the zonal polynomial of order 3 from the recurrence relation (3.16).

Table 3.2 is the coefficients of the zonal polynomials corresponding to $\kappa = (20, 0, 0)$. It can be written as follows

$$\frac{C_\kappa(Y)}{C_\kappa(I)} = 0.02439024(y_1^{20} + y_2^{20} + y_3^{20}) + \text{terms of lower weight.} \tag{3.17}$$

Table 3.2 The zonal polynomial in the case of $k = 20,\ \kappa = (20, 0, 0)$

k_1	k_2	k_3	$c_{(20,0,0),(k_1,k_2,k_3)}$	k_1	k_2	k_3	$c_{(20,0,0),(k_1,k_2,k_3)}$
20	0	0	0.0243902439024390243902439024390244	12	6	2	0.0026526214214431474802076283 4
19	1	0	0.012507817385866166353971232 02	12	5	3	0.0024114740194937704365523894 0
18	2	0	0.0096343998783023173267075706 1	12	4	4	0.0023444886300633879244259341 4
18	1	1	0.0064229332522015448844717137 4	11	9	0	0.0060686266474009895768725244 6
17	3	0	0.0082580570385448434228922033 8	11	8	1	0.0032128023427417003642266306 0
17	2	1	0.0049548342231269060537353220 3	11	7	2	0.0025702418741933602913813044 8
16	4	0	0.0074447635423245179342740318 4	11	6	3	0.0023066273229940412871370681 2
16	3	1	0.0042541505956140102481565896 2	11	5	4	0.0022017806264943121377217468 4
16	2	2	0.0038287355360526092233409306 6	10	10	0	0.0060397284252705086741255124 4
15	5	0	0.0069164254844821327905513586 1	10	9	1	0.0031788044343528993021713223 4
15	4	1	0.0038424586024900737725285325 6	10	8	2	0.0025243446978684788576066383 3
15	3	2	0.0032935359449914918050244564 8	10	7	3	0.0022438619536608700956503451 8
14	6	0	0.0065586793387330569565573228 2	10	6	4	0.0021144083794112045132089791 1
14	5	1	0.0035774614574907583399403579 0	10	5	5	0.0020759645906946371584233613 1
14	4	2	0.0029812178812422986166169649 2	9	9	2	0.0025095824481733415543457807 9
14	3	3	0.0028392551249926653491590142 1	9	8	3	0.0022143374542705954891286301 1
13	7	0	0.0063157652891503511433514960 5	9	7	4	0.0020667149573192224565200547 7
13	6	1	0.0034007966941578813848815747 9	9	6	5	0.0020031237278632463809348223 2
13	5	2	0.0027824700224928120421758339 2	8	8	4	0.0020515185238095222913985837 8
13	4	3	0.0025763611319377889279405869 7	8	7	5	0.0019694577828571413997426404 3
12	8	0	0.0061578711569215923647677086 5	8	6	6	0.0019442083241025626638485040 1
12	7	1	0.0032841979503581825945427779 4	7	7	6	0.0019256920543492049241928039 7

We know that

$$\sum_{\kappa} C_\kappa(Y) = (\mathrm{tr}\ Y)^k \,. \tag{3.18}$$

Thus if Y is the 3×3 unit matrix I, it may be written as

$$\left(\frac{1}{3}\right)^k \sum_{\kappa} C_\kappa(I) = 1. \tag{3.19}$$

So, we calculate it to check the correctness of the coefficients which we obtained. Table 3.3 is the results by quadruplex precision.

Table 3.3 The sum of the zonal polynomials $C_\kappa(I)$ of k

k	$\sum C_\kappa(I)$	$3^{-k} \sum C_\kappa(I)$
10	59049.0000000000000000000000000	1.000000000000000000000000000000
30	205891132094649.000000000000000	1.000000000000000000000000000000
50	717897987691852588770249.000000	1.000000000000000000000000000000
70	$2.5031555049932416013155719860 8\times10^{33}$	1.000000000000000000000000000000
90	$8.7279635680877124258913974794 8\times10^{42}$	1.000000000000000000000000000000
110	$3.0432527221704537086371993251 5\times10^{52}$	1.000000000000000000000000000000
130	$1.0611166119964724854368785575 3\times10^{62}$	1.000000000000000000000000000000
150	$3.6998848503512697292470078245 2\times10^{71}$	1.000000000000000000000000000000

3.3. Numerical analysis of type ${}_2F_1$ Gaussian

Let $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N$ be a sample from $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The maximum likelihood estimators of the $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are $\hat{\boldsymbol{\mu}} = \bar{\boldsymbol{X}} = N^{-1}\sum_{\alpha=1}^{N}\boldsymbol{X}_\alpha$ and $\hat{\boldsymbol{\Sigma}} = \boldsymbol{S} = N^{-1}\sum_{\alpha=1}^{N}(\boldsymbol{X}_\alpha - \bar{\boldsymbol{X}})(\boldsymbol{X}_\alpha - \bar{\boldsymbol{X}})'$. Let us partition $\boldsymbol{\Sigma}$ and $\boldsymbol{S}$ into p_1 and p_2 rows and columns as follows

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix} \qquad \boldsymbol{S} = \begin{pmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} \\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} \end{pmatrix}$$

where $p_1 \le p_2$. Now we consider the test of independence for the null hypothesis,

$$H : \boldsymbol{\Sigma}_{12} = \boldsymbol{0} \qquad (p_1 \times p_2),$$

against all alternatives

$$K : \boldsymbol{\Sigma}_{12} \ne \boldsymbol{0} \qquad (p_1 \times p_2).$$

Then the likelihood ratio criterion is given by the following

$$\lambda = \left(\frac{|\boldsymbol{S}|}{|\boldsymbol{S}_{11}| \cdot |\boldsymbol{S}_{22}|}\right)^{\frac{N}{2}} = |I - \boldsymbol{S}_{11}^{-1}\boldsymbol{S}_{12}\boldsymbol{S}_{22}^{-1}\boldsymbol{S}_{21}|^{\frac{N}{2}}. \tag{3.20}$$

Under alternatives K, the moments of the likelihood ratio criterion λ can be expressed as

$$E\left[\left(\frac{|\boldsymbol{S}|}{|\boldsymbol{S}_{11}| \cdot |\boldsymbol{S}_{22}|}\right)^h\right] = \frac{\Gamma_{p_1}(h + \frac{1}{2}(N - p_2 - 1))\Gamma_{p_1}(\frac{1}{2}(N-1))}{\Gamma_{p_1}(\frac{1}{2}(N - p_2 - 1))\Gamma_{p_1}(h + \frac{1}{2}(N-1))}\prod_{j=1}^{p_1}(1-\rho_j^2)^h$$
$$\cdot {}_2F_1(h, h; h + \tfrac{1}{2}(N-1); \boldsymbol{P}^2), \tag{3.21}$$

where $\rho_1, \rho_2, \cdots, \rho_{p_1}$ are the characteristic roots of $\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}$ and $\boldsymbol{P}^2 = diag(\rho_1^2, \rho_2^2, \cdots, \rho_{p_1}^2)$.

We compute the moments of the formula (3.20) by the quadruple precision arithmetic when $p_1 = 3$, $p_2 = 3$ and $\boldsymbol{P}^2 = diag(0.8, 0.6, 0.4)$. To see the accuracy of the numerical computation, we do the simulation study. With respect to the values obtained by the five million simulation, it may be expected for at most 3 figures to be reliable.

This is an example of the type ${}_2F_1$ Gaussian. As we have the program to compute zonal polynomials of order 3, we may compute any function which include the zonal polynomials.

Table 3.4 The values of moments of the likelihood ratio criterion λ and of the simulation given by the bracket.

N	$h=1$	$h=2$	$h=3$
10	$1.722472189 \times 10^{-2}$ (1.722×10^{-2})	$0.820016791 \times 10^{-3}$ (0.8222×10^{-3})	$0.781223473 \times 10^{-4}$ (0.7867×10^{-4})
15	$2.831553095 \times 10^{-2}$ (2.832×10^{-2})	$1.437775789 \times 10^{-3}$ (1.440×10^{-3})	$1.171515828 \times 10^{-4}$ (1.177×10^{-4})
20	$3.365802148 \times 10^{-2}$ (3.365×10^{-2})	$1.714080337 \times 10^{-3}$ (1.715×10^{-3})	$1.249567017 \times 10^{-4}$ (1.251×10^{-4})
25	$3.674212947 \times 10^{-2}$ (3.674×10^{-2})	$1.861615722 \times 10^{-3}$ (1.862×10^{-3})	$1.258496650 \times 10^{-4}$ (1.260×10^{-4})
30	$3.874093352 \times 10^{-2}$ (3.874×10^{-2})	$1.951669499 \times 10^{-3}$ (1.953×10^{-3})	$1.250932992 \times 10^{-4}$ (1.253×10^{-4})
35	$4.013932419 \times 10^{-2}$ (4.014×10^{-2})	$2.011871117 \times 10^{-3}$ (2.013×10^{-3})	$1.239848935 \times 10^{-4}$ (1.241×10^{-4})
40	$4.117175860 \times 10^{-2}$ (4.117×10^{-2})	$2.054780001 \times 10^{-3}$ (2.056×10^{-3})	$1.228811340 \times 10^{-4}$ (1.231×10^{-4})
45	$4.196497344 \times 10^{-2}$ (4.196×10^{-2})	$2.086838845 \times 10^{-3}$ (2.087×10^{-3})	$1.218781092 \times 10^{-4}$ (1.219×10^{-4})
50	$4.259334589 \times 10^{-2}$ (4.259×10^{-2})	$2.111667201 \times 10^{-3}$ (2.112×10^{-3})	$1.209927541 \times 10^{-4}$ (1.210×10^{-4})

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Appendix A

A.1 Recurrence relations of coefficients of multiple power series with the elementary symmetric functions for ${}_2F_1(a,b;c;Y)$

$j=4$

$$\begin{aligned}
&\gamma(j_1,j_2,j_3,j_4,\boldsymbol{o})\\
&\quad=\{2(j_1+1)(j_2+1)\,\gamma(j_1+1,j_2+1,j_3,j_4-1,\boldsymbol{o})\\
&\qquad+(j_2+1)(j_2+2)\,\gamma(j_1-1,j_2+2,j_3,j_4-1,\boldsymbol{o})\\
&\qquad-j_3(j_3+1)\,\gamma(j_1,j_2,j_3+1,j_4-1,\boldsymbol{o})\\
&\qquad-2(j_3+1)(j_4-1)\,\gamma(j_1,j_2,j_3+1,j_4-1,\boldsymbol{o})\\
&\qquad+(j_2+1)(j_2+2)\,\gamma(j_1,j_2+2,j_3,j_4-1,\boldsymbol{o})\\
&\qquad+2(j_1+1)(j_3+1)\,\gamma(j_1+1,j_2,j_3+1,j_4-1,\boldsymbol{o})\\
&\qquad+2(j_2+1)(j_3+1)\,\gamma(j_1-1,j_2+1,j_3+1,j_4-1,\boldsymbol{o})\\
&\qquad+(j_3+1)(j_3+2)\,\gamma(j_1,j_2-1,j_3+2,j_4-1,\boldsymbol{o})\\
&\qquad-(a+b-\tfrac{1}{2})(j_3+1)\,\gamma(j_1,j_2,j_3+1,j_4-1,\boldsymbol{o})\}\\
&\qquad\quad\cdot\{j_4(j_4+c-\tfrac{5}{2})\}^{-1}
\end{aligned}$$

$j=5$

$$\begin{aligned}
&\gamma(j_1,j_2,j_3,j_4,j_5,\boldsymbol{o})\\
&\quad=\{(j_2+1)(j_2+2)\,\gamma(j_1,j_2+2,j_3,j_4,j_5-1,\boldsymbol{o})\\
&\qquad+2(j_1+1)(j_3+1)\,\gamma(j_1+1,j_2,j_3+1,j_4,j_5-1,\boldsymbol{o})\\
&\qquad+2(j_2+1)(j_3+1)\,\gamma(j_1-1,j_2+1,j_3+1,j_4,j_5-1,\boldsymbol{o})\\
&\qquad+(j_3+1)(j_3+2)\,\gamma(j_1,j_2-1,j_3+2,j_4,j_5-1,\boldsymbol{o})\\
&\qquad-j_4(j_4+1)\,\gamma(j_1,j_2,j_3,j_4+1,j_5-1,\boldsymbol{o})\\
&\qquad-2(j_4+1)(j_5-1)\,\gamma(j_1,j_2,j_3,j_4+1,j_5-1,\boldsymbol{o})\\
&\qquad+2(j_2+1)(j_3+1)\,\gamma(j_1,j_2+1,j_3+1,j_4,j_5-1,\boldsymbol{o})\\
&\qquad+(j_3+1)(j_3+2)\,\gamma(j_1-1,j_2,j_3+2,j_4,j_5-1,\boldsymbol{o})\\
&\qquad+2(j_1+1)(j_4+1)\,\gamma(j_1+1,j_2,j_3,j_4+1,j_5-1,\boldsymbol{o})\\
&\qquad+2(j_2+1)(j_4+1)\,\gamma(j_1-1,j_2+1,j_3,j_4+1,j_5-1,\boldsymbol{o})\\
&\qquad+2(j_3+1)(j_4+1)\,\gamma(j_1,j_2-1,j_3+1,j_4+1,j_5-1,\boldsymbol{o})\\
&\qquad+(j_4+1)(j_4+2)\,\gamma(j_1,j_2,j_3-1,j_4+2,j_5-1,\boldsymbol{o})\\
&\qquad-(a+b-1)(j_4+1)\,\gamma(j_1,j_2,j_3,j_4+1,j_5-1,\boldsymbol{o})\}\\
&\qquad\quad\cdot\{j_5(j_5+c-3)\}^{-1}
\end{aligned}$$

$j=6$

$$\begin{aligned}
&\gamma(j_1,\dots,j_6,\boldsymbol{o})\\
&\quad=\{2(j_2+1)(j_3+1)\,\gamma(j_1,j_2+1,j_3+1,j_4,j_5,j_6-1,\boldsymbol{o})\\
&\qquad+(j_3+1)(j_3+2)\,\gamma(j_1-1,j_2,j_3+2,j_4,j_5,j_6-1,\boldsymbol{o})\\
&\qquad+2(j_1+1)(j_4+1)\,\gamma(j_1+1,j_2,j_3,j_4+1,j_5,j_6-1,\boldsymbol{o})\\
&\qquad+2(j_2+1)(j_4+1)\,\gamma(j_1-1,j_2+1,j_3,j_4+1,j_5,j_6-1,\boldsymbol{o})\\
&\qquad+2(j_3+1)(j_4+1)\,\gamma(j_1,j_2-1,j_3+1,j_4+1,j_5,j_6-1,\boldsymbol{o})\\
&\qquad+(j_4+1)(j_4+2)\,\gamma(j_1,j_2,j_3-1,j_4+2,j_5,j_6-1,\boldsymbol{o})\\
&\qquad-j_5(j_5+1)\,\gamma(j_1,j_2,j_3,j_4,j_5+1,j_6-1,\boldsymbol{o})\\
&\qquad-2(j_5+1)(j_6-1)\,\gamma(j_1,j_2,j_3,j_4,j_5+1,j_6-1,\boldsymbol{o})
\end{aligned}$$

$$
\begin{aligned}
&+ (j_3+1)(j_3+2)\,\gamma(j_1,j_2,j_3+2,j_4,j_5,j_6-1,\boldsymbol{o})\\
&+ 2(j_2+1)(j_4+1)\,\gamma(j_1,j_2+1,j_3,j_4+1,j_5,j_6-1,\boldsymbol{o})\\
&+ 2(j_3+1)(j_4+1)\,\gamma(j_1-1,j_2,j_3+1,j_4+1,j_5,j_6-1,\boldsymbol{o})\\
&+ (j_4+1)(j_4+2)\,\gamma(j_1,j_2-1,j_3,j_4+2,j_5,j_6-1,\boldsymbol{o})\\
&+ 2(j_1+1)(j_5+1)\,\gamma(j_1+1,j_2,j_3,j_4,j_5+1,j_6-1,\boldsymbol{o})\\
&+ 2(j_2+1)(j_5+1)\,\gamma(j_1-1,j_2+1,j_3,j_4,j_5+1,j_6-1,\boldsymbol{o})\\
&+ 2(j_3+1)(j_5+1)\,\gamma(j_1,j_2-1,j_3+1,j_4,j_5+1,j_6-1,\boldsymbol{o})\\
&+ 2(j_4+1)(j_5+1)\,\gamma(j_1,j_2,j_3-1,j_4+1,j_5+1,j_6-1,\boldsymbol{o})\\
&+ (j_5+1)(j_5+2)\,\gamma(j_1,j_2,j_3,j_4-1,j_5+2,j_6-1,\boldsymbol{o})\\
&- (a+b-\tfrac{3}{2})(j_5+1)\,\gamma(j_1,j_2,j_3,j_4,j_5+1,j_6-1,\boldsymbol{o})\}\\
&\qquad\cdot\{j_6(j_6+c-\tfrac{7}{2})\}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
j=7 \qquad &\gamma(j_1,\ldots,j_7,\boldsymbol{o})\\
&= \{(j_3+1)(j_3+2)\,\gamma(j_1,j_2,j_3+2,j_4,j_5,j_6,j_7-1,\boldsymbol{o})\\
&\quad + 2(j_2+1)(j_4+1)\,\gamma(j_1,j_2+1,j_3,j_4+1,j_5,j_6,j_7-1,\boldsymbol{o})\\
&\quad + 2(j_3+1)(j_4+1)\,\gamma(j_1-1,j_2,j_3+1,j_4+1,j_5,j_6,j_7-1,\boldsymbol{o})\\
&\quad + (j_4+1)(j_4+2)\,\gamma(j_1,j_2-1,j_3,j_4+2,j_5,j_6,j_7-1,\boldsymbol{o})\\
&\quad + 2(j_1+1)(j_5+1)\,\gamma(j_1+1,j_2,j_3,j_4,j_5+1,j_6,j_7-1,\boldsymbol{o})\\
&\quad + 2(j_2+1)(j_5+1)\,\gamma(j_1-1,j_2+1,j_3,j_4,j_5+1,j_6,j_7-1,\boldsymbol{o})\\
&\quad + 2(j_3+1)(j_5+1)\,\gamma(j_1,j_2-1,j_3+1,j_4,j_5+1,j_6,j_7-1,\boldsymbol{o})\\
&\quad + 2(j_4+1)(j_5+1)\,\gamma(j_1,j_2,j_3-1,j_4+1,j_5+1,j_6,j_7-1,\boldsymbol{o})\\
&\quad + (j_5+1)(j_5+2)\,\gamma(j_1,j_2,j_3,j_4-1,j_5+2,j_6,j_7-1,\boldsymbol{o})\\
&\quad - j_6(j_6+1)\,\gamma(j_1,j_2,j_3,j_4,j_5,j_6+1,j_7-1,\boldsymbol{o})\\
&\quad - (j_6+1)(j_7-1)\,\gamma(j_1,j_2,j_3,j_4,j_5,j_6+1,j_7-1,\boldsymbol{o})\\
&\quad + 2(j_3+1)(j_4+1)\,\gamma(j_1,j_2,j_3+1,j_4+1,j_5,j_6,j_7-1,\boldsymbol{o})\\
&\quad + (j_4+1)(j_4+2)\,\gamma(j_1-1,j_2,j_3,j_4+2,j_5,j_6,j_7-1,\boldsymbol{o})\\
&\quad + 2(j_2+1)(j_5+1)\,\gamma(j_1,j_2+1,j_3,j_4,j_5+1,j_6,j_7-1,\boldsymbol{o})\\
&\quad + 2(j_3+1)(j_5+1)\,\gamma(j_1-1,j_2,j_3+1,j_4,j_5+1,j_6,j_7-1,\boldsymbol{o})\\
&\quad + 2(j_4+1)(j_5+1)\,\gamma(j_1,j_2-1,j_3,j_4+1,j_5+1,j_6,j_7-1,\boldsymbol{o})\\
&\quad + (j_5+1)(j_5+2)\,\gamma(j_1,j_2,j_3-1,j_4,j_5+2,j_6,j_7-1,\boldsymbol{o})\\
&\quad + 2(j_1+1)(j_6+1)\,\gamma(j_1+1,j_2,j_3,j_4,j_5,j_6+1,j_7-1,\boldsymbol{o})\\
&\quad + 2(j_2+1)(j_6+1)\,\gamma(j_1-1,j_2+1,j_3,j_4,j_5,j_6+1,j_7-1,\boldsymbol{o})\\
&\quad + 2(j_3+1)(j_6+1)\,\gamma(j_1,j_2-1,j_3+1,j_4,j_5,j_6+1,j_7-1,\boldsymbol{o})\\
&\quad + 2(j_4+1)(j_6+1)\,\gamma(j_1,j_2,j_3-1,j_4+1,j_5,j_6+1,j_7-1,\boldsymbol{o})\\
&\quad + 2(j_5+1)(j_6+1)\,\gamma(j_1,j_2,j_3,j_4-1,j_5+1,j_6+1,j_7-1,\boldsymbol{o})\\
&\quad + (j_6+1)(j_6+2)\,\gamma(j_1,j_2,j_3,j_4,j_5-1,j_6+2,j_7-1,\boldsymbol{o})\\
&\quad - (a+b-2)(j_6+1)\,\gamma(j_1,j_2,j_3,j_4,j_5,j_6+1,j_7-1,\boldsymbol{o})\}\\
&\qquad\cdot\{j_7(j_7+c-4)\}^{-1}
\end{aligned}
$$

A.2 Recurrence relations of coefficients of multiple power series with the elementary symmetric functions for ${}_1F_1(a;c;Y)$

$j=1$

$$\gamma(j_1,\boldsymbol{o}) = \{(j_1+a-1)\,\gamma(j_1-1,\boldsymbol{o})\}\cdot\{j_1(c+j_1-1)\}^{-1}$$

$j=2$

$$\gamma(j_1,j_2,\boldsymbol{o}) = \{(j_1+1)(j_1+2)\,\gamma(j_1+2,j_2-1,\boldsymbol{o}) - (j_1+1)\,\gamma(j_1+1,j_2-1,\boldsymbol{o})\} \cdot\{j_2(j_2+c-\tfrac{3}{2})\}^{-1}$$

$j=3$

$$\gamma(j_1,j_2,j_3,\boldsymbol{o}) = \{2(j_1+1)(j_2+1)\,\gamma(j_1+1,j_2+1,j_3-1,\boldsymbol{o}) + (j_2+1)(j_2+2)\,\gamma(j_1-1,j_2+2,j_3-1,\boldsymbol{o}) - (j_2+1)\,\gamma(j_1,j_2+1,j_3-1,\boldsymbol{o})\} \cdot\{j_3(j_3+c-2)\}^{-1}$$

$j=4$

$$\gamma(j_1,j_2,j_3,j_4,\boldsymbol{o}) = \{(j_2+1)(j_2+2)\,\gamma(j_1,j_2+2,j_3,j_4-1,\boldsymbol{o}) + 2(j_1+1)(j_3+1)\,\gamma(j_1+1,j_2,j_3+1,j_4-1,\boldsymbol{o}) + 2(j_2+1)(j_3+1)\,\gamma(j_1-1,j_2+1,j_3+1,j_4-1,\boldsymbol{o}) + (j_3+1)(j_3+2)\,\gamma(j_1,j_2-1,j_3+2,j_4-1,\boldsymbol{o}) - (j_3+1)\,\gamma(j_1,j_2,j_3+1,j_4-1,\boldsymbol{o})\} \cdot\{j_4(j_4+c-\tfrac{5}{2})\}^{-1}$$

$j=5$

$$\gamma(j_1,j_2,j_3,j_4,j_5,\boldsymbol{o}) = \{2(j_2+1)(j_3+1)\,\gamma(j_1,j_2+1,j_3+1,j_4,j_5-1,\boldsymbol{o}) + (j_3+1)(j_3+2)\,\gamma(j_1-1,j_2,j_3+2,j_4,j_5-1,\boldsymbol{o}) + 2(j_1+1)(j_4+1)\,\gamma(j_1+1,j_2,j_3,j_4+1,j_5-1,\boldsymbol{o}) + 2(j_2+1)(j_4+1)\,\gamma(j_1-1,j_2+1,j_3,j_4+1,j_5-1,\boldsymbol{o}) + 2(j_3+1)(j_4+1)\,\gamma(j_1,j_2-1,j_3+1,j_4+1,j_5-1,\boldsymbol{o}) + (j_4+1)(j_4+2)\,\gamma(j_1,j_2,j_3-1,j_4+2,j_5-1,\boldsymbol{o}) - (j_4+1)\,\gamma(j_1,j_2,j_3,j_4+1,j_5-1,\boldsymbol{o})\} \cdot\{j_5(j_5+c-3)\}^{-1}$$

$j=6$

$$\gamma(j_1,\ldots,j_6,\boldsymbol{o}) = \{(j_3+1)(j_3+2)\,\gamma(j_1,j_2,j_3+2,j_4,j_5,j_6-1,\boldsymbol{o}) + 2(j_2+1)(j_4+1)\,\gamma(j_1,j_2+1,j_3,j_4+1,j_5,j_6-1,\boldsymbol{o}) + 2(j_3+1)(j_4+1)\,\gamma(j_1-1,j_2,j_3+1,j_4+1,j_5,j_6-1,\boldsymbol{o}) + (j_4+1)(j_4+2)\,\gamma(j_1,j_2-1,j_3,j_4+2,j_5,j_6-1,\boldsymbol{o}) + 2(j_1+1)(j_5+1)\,\gamma(j_1+1,j_2,j_3,j_4,j_5+1,j_6-1,\boldsymbol{o})$$

$$
\begin{aligned}
&+ 2(j_2+1)(j_5+1)\,\gamma(j_1-1, j_2+1, j_3, j_4, j_5+1, j_6-1, \boldsymbol{o}) \\
&+ 2(j_3+1)(j_5+1)\,\gamma(j_1, j_2-1, j_3+1, j_4, j_5+1, j_6-1, \boldsymbol{o}) \\
&+ 2(j_4+1)(j_5+1)\,\gamma(j_1, j_2, j_3-1, j_4+1, j_5+1, j_6-1, \boldsymbol{o}) \\
&+ (j_5+1)(j_5+2)\,\gamma(j_1, j_2, j_3, j_4-1, j_5+2, j_6-1, \boldsymbol{o}) \\
&- (j_5+1)\,\gamma(j_1, j_2, j_3, j_4, j_5+1, j_6-1, \boldsymbol{o})\} \\
&\qquad \cdot \{j_6(j_6+c-\tfrac{7}{2})\}^{-1}
\end{aligned}
$$

$j = 7$

$$
\begin{aligned}
&\gamma(j_1, \ldots, j_7, \boldsymbol{o}) \\
&\quad = \{2(j_3+1)(j_4+1)\,\gamma(j_1, j_2, j_3+1, j_4+1, j_5, j_6, j_7-1, \boldsymbol{o}) \\
&\qquad + (j_4+1)(j_4+2)\,\gamma(j_1-1, j_2, j_3, j_4+2, j_5, j_6, j_7-1, \boldsymbol{o}) \\
&\qquad + 2(j_2+1)(j_5+1)\,\gamma(j_1, j_2+1, j_3, j_4, j_5+1, j_6, j_7-1, \boldsymbol{o}) \\
&\qquad + 2(j_3+1)(j_5+1)\,\gamma(j_1-1, j_2, j_3+1, j_4, j_5+1, j_6, j_7-1, \boldsymbol{o}) \\
&\qquad + 2(j_4+1)(j_5+1)\,\gamma(j_1, j_2-1, j_3, j_4+1, j_5+1, j_6, j_7-1, \boldsymbol{o}) \\
&\qquad + (j_5+1)(j_5+2)\,\gamma(j_1, j_2, j_3-1, j_4, j_5+2, j_6, j_7-1, \boldsymbol{o}) \\
&\qquad + 2(j_1+1)(j_6+1)\,\gamma(j_1+1, j_2, j_3, j_4, j_5, j_6+1, j_7-1, \boldsymbol{o}) \\
&\qquad + 2(j_2+1)(j_6+1)\,\gamma(j_1-1, j_2+1, j_3, j_4, j_5, j_6+1, j_7-1, \boldsymbol{o}) \\
&\qquad + 2(j_3+1)(j_6+1)\,\gamma(j_1, j_2-1, j_3+1, j_4, j_5, j_6+1, j_7-1, \boldsymbol{o}) \\
&\qquad + 2(j_4+1)(j_6+1)\,\gamma(j_1, j_2, j_3-1, j_4+1, j_5, j_6+1, j_7-1, \boldsymbol{o}) \\
&\qquad + 2(j_5+1)(j_6+1)\,\gamma(j_1, j_2, j_3, j_4-1, j_5+1, j_6+1, j_7-1, \boldsymbol{o}) \\
&\qquad + (j_6+1)(j_6+2)\,\gamma(j_1, j_2, j_3, j_4, j_5-1, j_6+2, j_7-1, \boldsymbol{o}) \\
&\qquad - (j_6+1)\,\gamma(j_1, j_2, j_3, j_4, j_5, j_6+1, j_7-1, \boldsymbol{o})\} \\
&\qquad\qquad \cdot \{j_7(j_7+c-4)\}^{-1}
\end{aligned}
$$

A.3 Recurrence relations of coefficients of multiple power series with the elementary symmetric functions for ${}_0F_1(c;Y)$

$j = 1$

$$
\begin{aligned}
&\gamma(j_1, \boldsymbol{o}) \\
&\quad = \gamma(j_1-1, \boldsymbol{o}) \cdot \{j_1(c+j_1-1)\}^{-1}
\end{aligned}
$$

$j = 2$

$$
\begin{aligned}
&\gamma(j_1, j_2, \boldsymbol{o}) \\
&\quad = \{(j_1+1)(j_1+2)\,\gamma(j_1+2, j_2-1, \boldsymbol{o})\} \\
&\qquad\qquad \cdot \{j_2(j_2+c-\tfrac{3}{2})\}^{-1}
\end{aligned}
$$

$j = 3$

$$
\begin{aligned}
&\gamma(j_1, j_2, j_3, \boldsymbol{o}) \\
&\quad = \{2(j_1+1)(j_2+1)\,\gamma(j_1+1, j_2+1, j_3-1, \boldsymbol{o}) \\
&\qquad + (j_2+1)(j_2+2)\,\gamma(j_1-1, j_2+2, j_3-1, \boldsymbol{o})\} \\
&\qquad\qquad \cdot \{j_3(j_3+c-2)\}^{-1}
\end{aligned}
$$

$j = 4$

$$
\begin{aligned}
&\gamma(j_1, j_2, j_3, j_4, \boldsymbol{o}) \\
&\quad = \{(j_2+1)(j_2+2)\,\gamma(j_1, j_2+2, j_3, j_4-1, \boldsymbol{o}) \\
&\qquad + 2(j_1+1)(j_3+1)\,\gamma(j_1+1, j_2, j_3+1, j_4-1, \boldsymbol{o})
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(j_2+1)(j_3+1)\,\gamma(j_1-1, j_2+1, j_3+1, j_4-1, \boldsymbol{o})\\
&+ (j_3+1)(j_3+2)\,\gamma(j_1, j_2-1, j_3+2, j_4-1, \boldsymbol{o})\}\\
&\qquad \cdot \{j_4(j_4+c-\tfrac{5}{2})\}^{-1}
\end{aligned}
$$

$j=5$

$$
\begin{aligned}
&\gamma(j_1, j_2, j_3, j_4, j_5, \boldsymbol{o})\\
&= \{2(j_2+1)(j_3+1)\,\gamma(j_1, j_2+1, j_3+1, j_4, j_5-1, \boldsymbol{o})\\
&\quad + (j_3+1)(j_3+2)\,\gamma(j_1-1, j_2, j_3+2, j_4, j_5-1, \boldsymbol{o})\\
&\quad + 2(j_1+1)(j_4+1)\,\gamma(j_1+1, j_2, j_3, j_4+1, j_5-1, \boldsymbol{o})\\
&\quad + 2(j_2+1)(j_4+1)\,\gamma(j_1-1, j_2+1, j_3, j_4+1, j_5-1, \boldsymbol{o})\\
&\quad + 2(j_3+1)(j_4+1)\,\gamma(j_1, j_2-1, j_3+1, j_4+1, j_5-1, \boldsymbol{o})\\
&\quad + (j_4+1)(j_4+2)\,\gamma(j_1, j_2, j_3-1, j_4+2, j_5-1, \boldsymbol{o})\}\\
&\qquad \cdot \{j_5(j_5+c-3)\}^{-1}
\end{aligned}
$$

$j=6$

$$
\begin{aligned}
&\gamma(j_1, \ldots, j_6, \boldsymbol{o})\\
&= \{(j_3+1)(j_3+2)\,\gamma(j_1, j_2, j_3+2, j_4, j_5, j_6-1, \boldsymbol{o})\\
&\quad + 2(j_2+1)(j_4+1)\,\gamma(j_1, j_2+1, j_3, j_4+1, j_5, j_6-1, \boldsymbol{o})\\
&\quad + 2(j_3+1)(j_4+1)\,\gamma(j_1-1, j_2, j_3+1, j_4+1, j_5, j_6-1, \boldsymbol{o})\\
&\quad + (j_4+1)(j_4+2)\,\gamma(j_1, j_2-1, j_3, j_4+2, j_5, j_6-1, \boldsymbol{o})\\
&\quad + 2(j_1+1)(j_5+1)\,\gamma(j_1+1, j_2, j_3, j_4, j_5+1, j_6-1, \boldsymbol{o})\\
&\quad + 2(j_2+1)(j_5+1)\,\gamma(j_1-1, j_2+1, j_3, j_4, j_5+1, j_6-1, \boldsymbol{o})\\
&\quad + 2(j_3+1)(j_5+1)\,\gamma(j_1, j_2-1, j_3+1, j_4, j_5+1, j_6-1, \boldsymbol{o})\\
&\quad + 2(j_4+1)(j_5+1)\,\gamma(j_1, j_2, j_3-1, j_4+1, j_5+1, j_6-1, \boldsymbol{o})\\
&\quad + (j_5+1)(j_5+2)\,\gamma(j_1, j_2, j_3, j_4-1, j_5+2, j_6-1, \boldsymbol{o})\}\\
&\qquad \cdot \{j_6(j_6+c-\tfrac{7}{2})\}^{-1}
\end{aligned}
$$

$j=7$

$$
\begin{aligned}
&\gamma(j_1, \ldots, j_7, \boldsymbol{o})\\
&= \{2(j_3+1)(j_4+1)\,\gamma(j_1, j_2, j_3+1, j_4+1, j_5, j_6, j_7-1, \boldsymbol{o})\\
&\quad + (j_4+1)(j_4+2)\,\gamma(j_1-1, j_2, j_3, j_4+2, j_5, j_6, j_7-1, \boldsymbol{o})\\
&\quad + 2(j_2+1)(j_5+1)\,\gamma(j_1, j_2+1, j_3, j_4, j_5+1, j_6, j_7-1, \boldsymbol{o})\\
&\quad + 2(j_3+1)(j_5+1)\,\gamma(j_1-1, j_2, j_3+1, j_4, j_5+1, j_6, j_7-1, \boldsymbol{o})\\
&\quad + 2(j_4+1)(j_5+1)\,\gamma(j_1, j_2-1, j_3, j_4+1, j_5+1, j_6, j_7-1, \boldsymbol{o})\\
&\quad + (j_5+1)(j_5+2)\,\gamma(j_1, j_2, j_3-1, j_4, j_5+2, j_6, j_7-1, \boldsymbol{o})\\
&\quad + 2(j_1+1)(j_6+1)\,\gamma(j_1+1, j_2, j_3, j_4, j_5, j_6+1, j_7-1, \boldsymbol{o})\\
&\quad + 2(j_2+1)(j_6+1)\,\gamma(j_1-1, j_2+1, j_3, j_4, j_5, j_6+1, j_7-1, \boldsymbol{o})\\
&\quad + 2(j_3+1)(j_6+1)\,\gamma(j_1, j_2-1, j_3+1, j_4, j_5, j_6+1, j_7-1, \boldsymbol{o})\\
&\quad + 2(j_4+1)(j_6+1)\,\gamma(j_1, j_2, j_3-1, j_4+1, j_5, j_6+1, j_7-1, \boldsymbol{o})\\
&\quad + 2(j_5+1)(j_6+1)\,\gamma(j_1, j_2, j_3, j_4-1, j_5+1, j_6+1, j_7-1, \boldsymbol{o})\\
&\quad + (j_6+1)(j_6+2)\,\gamma(j_1, j_2, j_3, j_4, j_5-1, j_6+2, j_7-1, \boldsymbol{o})\}\\
&\qquad \cdot \{j_7(j_7+c-4)\}^{-1}
\end{aligned}
$$